Inclusive electromagnetic decays of the heavy quarkonium at next to leading log accuracy*

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We show that perturbation theory may give reasonable numbers for the decays of the bottomonium and charmonium ground states to e^+e^- and to $\gamma\gamma$. To reach this conclusion it is important to perform the resummation of logs. In particular, we obtain the value $\Gamma(\eta_b(1S) \to \gamma\gamma) = 0.35 \pm 0.1(\mathrm{th.}) \pm 0.05(\Lambda_{\rm QCD})$ KeV.

1. Introduction

In Ref. [1], the decays of Heavy Quarkonium to e^+e^- and $\gamma\gamma$ were computed with next to leading log (NLL) accuracy within perturbation theory. When presenting these results in this conference, one (reasonable) complaint was the complete absence of numbers. We would like to fill this gap by doing the phenomenological analysis of these results that we quote here for ease of reference.

We first quote the matching coefficients at the hard scale [2, 3]:

$$b_1(m) = 1 - 2C_f \frac{\alpha_{\rm s}(m)}{\pi}, \tag{1}$$

$$b_0(m) = 1 + \left(\frac{\pi^2}{4} - 5\right) \frac{C_f}{2} \frac{\alpha_{\rm s}(m)}{\pi}, \tag{2}$$

where $C_f = (N_c^2 - 1)/(2N_c)$, whereas the renormalization group improved matching coefficients at NLL (for the vector and pseudo-scalar) read ([1, 4])

$$B_s(\nu_p) = b_s(m) + A_1 \frac{\alpha_{\rm s}(m)}{w^{\beta_0}} \ln(w^{\beta_0}) + A_2 \alpha_{\rm s}(m) \left[z^{\beta_0} - 1\right] \tag{3}$$

$$+ A_3 \alpha_{\rm s}(m) \left[z^{\beta_0 - 2C_A} - 1\right] + A_4 \alpha_{\rm s}(m) \left[z^{\beta_0 - 13C_A/6} - 1\right] + A_5 \alpha_{\rm s}(m) \ln(z^{\beta_0}),$$

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where $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$, $z = \left[\frac{\alpha_s(\nu_p)}{\alpha_s(m)}\right]^{\frac{1}{\beta_0}}$ and $w = \left[\frac{\alpha_s(\nu_p^2/m)}{\alpha_s(\nu_p)}\right]^{\frac{1}{\beta_0}}$. The coefficients A_i in Eq. (4) read

$$
\begin{aligned}
A_1 &= \frac{8\pi C_f}{3\beta_0^2}\left(C_A^2 + 2C_f^2 + 3C_f C_A\right) , \\
A_2 &= \frac{\pi C_f[3\beta_0(26C_A^2 + 19C_A C_f - 32C_f^2) - C_A(208C_A^2 + 651C_A C_f + 116C_f^2)]}{78\,\beta_0^2\,C_A} , \\
A_3 &= -\frac{\pi C_f^2\Big[\beta_0(4s(s+1)-3) + C_A(15-14s(s+1))\Big]}{6(\beta_0 - 2C_A)^2} , \\
A_4 &= \frac{24\pi C_f^2(3\beta_0 - 11C_A)(5C_A + 8C_f)}{13\,C_A(6\beta_0 - 13C_A)^2} , \\
A_5 &= \frac{-\pi C_f^2}{\beta_0^2\,(6\beta_0 - 13C_A)(\beta_0 - 2C_A)}\Bigg\{C_A^2(-9C_A + 100C_f) \\
&\quad + \beta_0 C_A(-74C_f + C_A(42 - 13s(s+1))) + 6\beta_0^2(2C_f + C_A(-3 + s(s+1)))\Bigg\} .
\end{aligned}
\tag{4}
$$

By setting $\nu_p \sim mC_f\alpha_s/n$, $B_s(\nu_p)$ includes all the large logs at NLL order in any (inclusive enough) S-wave heavy-quarkonium production observable we can think of. For instance, the decays to e^+e^- and to two photons at NLL order read

$$
\begin{aligned}
\Gamma(V_Q(nS)\to e^+e^-) &= 2\frac{C_A}{3}\left[\frac{\alpha_{em}Q}{M_{V_Q(nS)}}\right]^2\left(\frac{m_Q C_f\alpha_s}{n}\right)^3\{B_1(\nu_p)(1+\delta\phi_n)\}^2 \qquad (5)\\
&\simeq 2\frac{C_A}{3}\left[\frac{\alpha_{em}Q}{M_{V_Q(nS)}}\right]^2\left(\frac{m_Q C_f\alpha_s}{n}\right)^3\{1 + 2(B_1(\nu_p)-1) + 2\delta\phi_n\} , \\
\Gamma(P_Q(nS)\to\gamma\gamma) &= 2C_A\left[\frac{\alpha_{em}Q^2}{M_{P_Q(nS)}}\right]^2\left(\frac{m_Q C_f\alpha_s}{n}\right)^3\{B_0(\nu_p)(1+\delta\phi_n)\}^2 \qquad (6)\\
&\simeq 2C_A\left[\frac{\alpha_{em}Q^2}{M_{P_Q(nS)}}\right]^2\left(\frac{m_Q C_f\alpha_s}{n}\right)^3\{1 + 2(B_0(\nu_p)-1) + 2\delta\phi_n\} ,
\end{aligned}
$$

where V and P stand for the vector and pseudo-scalar heavy quarkonium, $\alpha_s = \alpha_s(\nu_p)$, and ($\Psi_n(z) = \frac{d^n \ln\Gamma(z)}{dz^n}$ and $\Gamma(z)$ is the Euler Γ-function)

$$
\delta\phi_n = \frac{\alpha_s}{\pi}\left[-C_A + \frac{\beta_0}{4}\left(3\log\left(\frac{\nu_p n}{mC_f\alpha_s}\right) + \Psi_1(n+1) - 2n\Psi_2(n) + \frac{3}{2} + \gamma_E + \frac{2}{n}\right)\right] ,
\tag{7}
$$

which has been read from Ref. [5].

It is not our aim to perform a full fledge analysis of Eqs. (5) and (6) here but rather to see what are the general trends obtained by the introduction of the resummation of logs, as well as to give some predictions when the results appear to be reliable enough.

2. Phenomenological analysis

In this section we perform the phenomenological analysis of the above results for the bottomonium and charmonium systems.

2.1. b-$\bar{b}$ 1S states

We first consider the b-$\bar{b}$ 1S states and their decays $\Gamma(\Upsilon(1S) \to e^+e^-)$ and $\Gamma(\eta_b(1S) \to \gamma\gamma)$. For the first decay we will be able to compare our results with experiment whereas our numbers for the second can be considered to be a prediction.

We plot $\Gamma(\Upsilon(1S) \to e^+e^-)$ in Fig. 1 versus the renormalization scale ν. We consider the LO/LL result (they are equal), the NLO result and the NLL result. We can see that the LO/LL result can match the experimental figure for a reasonable value of ν, close to the soft scale. This agreement is destroyed once the NLO is considered. The best value we can obtain is 0.667 KeV. The reason seems to be the fact that we now have two scales in the problem: the hard ($\sim m$) and the soft ($\sim m\alpha_s$) scale. The final outcome is that perturbation theory breaks down before the normalization scale ν can reach the typical soft scale of the problem. The standard solution to this problem goes by doing a renormalization group analysis, summing up all the large logs that appear in the problem. Actually, in our case, we are going to have large logs produced by the ratio of the hard and soft scale and by the ratio of the soft and ultrasoft scale ($\sim m\alpha_s^2$). We can see that the use of the NLL expressions improves the agreement with the experimental result (the best value now becomes 0.837 KeV) and enlarges the range of applicability of perturbation theory. Moreover, the expansion seems to be convergent, being the NLL result a correction with respect the LL order one. Nevertheless, perturbation theory still seems to break down before the renormalization scale ν can reach the typical soft scale (although getting closer to it than in the NLO calculation) and the optimal result is off the experimental value by around 50%. This is a large effect. Therefore, in order to confirm this picture, a full NNLL result should be obtained. This is a difficult computation but were the convergent pattern confirmed the outcomes will be important. One could then start quantifying the size of the non-perturbative effects in a reliable manner.

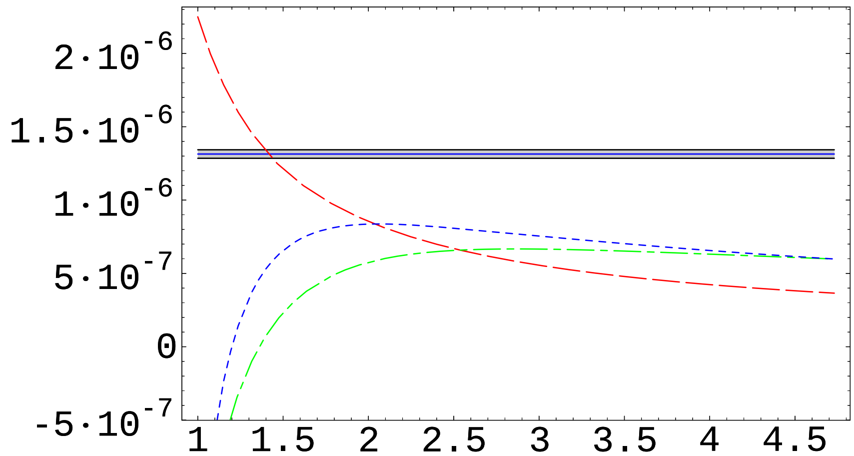

Fig. 1. *Plot of* $\Gamma(\Upsilon(1S) \to e^+e^-)$ *with LO/LL (dashed line), NLO (dot-dashed line) and NLL (dotted line) accuracy versus the renormalization scale* ν. *The horizontal line and its band give the experimental value and its errors:* $\Gamma(\Upsilon(1S) \to e^+e^-) = 1.314 \pm 0.029$ *KeV [6].*

We now perform a similar analysis for $\Gamma(\eta_b(1S) \to \gamma\gamma)$. For this decay no experimental figure exists. Therefore, we will be able to give a prediction for it. The picture is completely analogous to the $\Gamma(\Upsilon(1S) \to e^+e^-)$ case. Actually the scale of minimal sensitive is a little bit smaller than in the previous case, which makes us more confident about the result. This confidence will be further boost below when we consider the charmonium case for which experimental numbers exist. We anticipate that very good agreement is reached in that case. We then give a prediction for this decay:

$$\Gamma(\eta_b(1S) \to \gamma\gamma) = 0.35 \pm 0.1(\mathrm{th.}) \pm 0.05(\Lambda_{\mathrm{QCD}})\mathrm{KeV}, \tag{8}$$

where "Λ_{QCD}" stands for the error due to the variation of α_{s} ($\alpha_s(M_Z) = 0.118 \pm 0.003$) and "th" for the theoretical errors. The later are difficult to obtain and here we only pretend to roughly estimate their size. We consider the variation of the decay if we increase ν by two GeV with respect the optimal scale. This gives the theoretical error quoted in Eq. (8). We consider this estimate conservative in view of the good agreement with data obtained for the η_c case. Note that to have larger errors would be inconsistent with the assumption that the remaining corrections (perturbative and non-perturbative) are smaller, or at least of the same order, than the difference between the LL and NLL result.

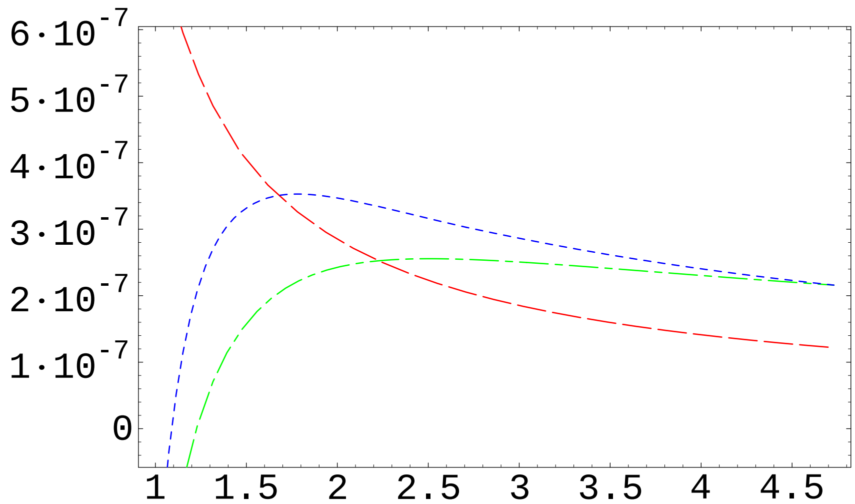

Fig. 2. *Plot of* $\Gamma(\eta_b(1S) \to \gamma\gamma)$ *with LO/LL (dashed line), NLO (dot-dashed line) and NLL (dotted line) accuracy versus the renormalization scale* ν.

For the above states one may consider reasonable to believe that a perturbative approach is a good starting point for these systems, since the soft scale is clearly in the perturbative regime. Nevertheless, one should be careful since the ultrasoft scale also enters the game through the matching coefficient (ultrasoft effects first appear at $O(\alpha_s^3 \ln^2 \alpha_s)$). This means that for $\nu \lesssim 1.5$ GeV, $\alpha_s(\nu^2/m_b)$ starts to blow up. Obviously, the situation is even worse for the $n = 2$ bottomonium states, since the soft scale goes divided by $1/n^2$ (actually, we anticipate that a bad behavior is found in this case). Surprisingly, however, for the charmonium system we are in a situation quite similar to the $n = 1$ bottomonium states since, although the typical soft scale is smaller than the $n = 1$ bottomonium soft scale, this is compensated by the fact that the charm mass is smaller than the bottom one, so that we can run down ν to scales quite close to the typical soft scales of the problem before the break down of perturbation theory takes place. With all these qualifications let us consider the $n = 1$ charmonium or $n = 2$ bottomonium states and see whether reasonable numbers are obtained.

2.2. $n = 1$ *charmonium states*

We study $\Gamma(J/\Psi(1S) \to e^+e^-)$ in Fig. 3. Surprisingly a pretty similar picture than in the $\Gamma(\Upsilon(1S) \to e^+e^-)$ appears. Actually, the difference with experiment is of the same order, around 50 %.

We now consider $\Gamma(\eta_c(1S) \to \gamma\gamma)$ in Fig. 4. In this case we have exper-

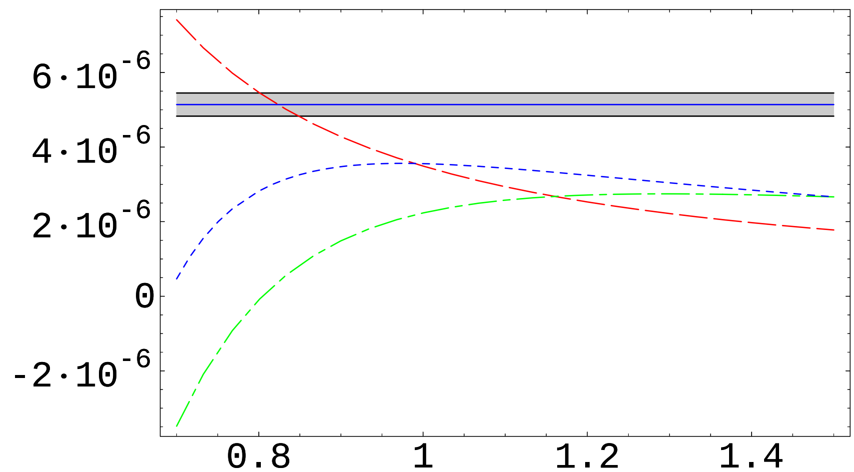

Fig. 3. *Plot of* $\Gamma(J/\Psi(1S) \rightarrow e^+e^-)$ *with LO/LL (dashed line), NLO (dot-dashed line) and NLL (dotted line) accuracy versus the renormalization scale* ν*. The horizontal line and its band give the experimental value and its errors:* $\Gamma(J/\Psi(1S) \rightarrow e^+e^-) = 5.14 \pm 0.31$ *KeV [6].*

imental data to compare with, unlike the η_b case. A similar pattern to the η_b case is found and, moreover, we get agreement with experiment. This is quite shocking since, at the scale of minimal sensitive, the perturbative result hits the experimental one just in the middle. Note that to get this agreement the resummation of logs is necessary. The scale of minimal sensitive is around 850 MeV, of the same order than the typical soft scale of the state. However, these numbers should be taken with caution at this stage, since the running involves the ultrasoft scale, which has been run down to very low scales $\sim$ 500 MeV in a correlated way.

2.3. $n = 2$ *bottomonium states*

We now turn to the $n = 2$ bottomonium states. We show our results in Figs. 5 and 6. In principle, this is the most problematic case since the soft scale of the $\Upsilon(2S)$ is $\sim 1/n^2 (= 1/4) \times$ the soft scale of the $\Upsilon(1S)$ (even if partially corrected by the fact that α_s would be larger for the 2S state than for the 1S state). Actually for $\Gamma(\Upsilon(2S) \rightarrow e^+e^-)$ we can compare with experiment and our result is a factor two smaller than the experimental number. We note that, even for the leading order result, we have to go to very small scales to get agreement with experiment ($\sim$ 600 MeV). This raises doubts about our perturbative analysis for the $n = 2$ bottomonium states.

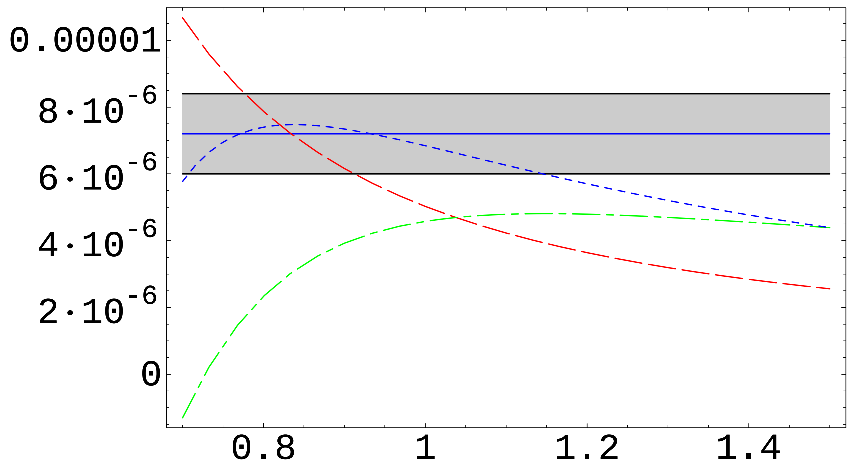

Fig. 4. *Plot of* $\Gamma(\eta_c(1S) \to \gamma\gamma)$ *with LO/LL (dashed line), NLO (dot-dashed line) and NLL (solid line) accuracy versus the renormalization scale* ν. *The horizontal line and its band give the experimental value and its errors:* $\Gamma(\eta_c(1S) \to \gamma\gamma) = 7.2 \pm 1.2$ *KeV [6].*

Somewhat, even if the resummation of logs helps, perturbation theory still breaks down before one can reach the typical soft scale of the problem (which is very small).

2.4. Final discussion

Even if one should wait for the complete NNLL result, we can start to see some trends. The uncalculated (perturbative or non-perturbative) terms seems to be larger for the decays into e^+e^- than for the decays into two photons. One could then start to speculate about the size of the non-perturbative effects in each case. Good enough, the agreement obtained with the experimental figure of $\Gamma(\eta_c(1S) \to \gamma\gamma)$ makes us quite confident that the result we have obtained for $\Gamma(\eta_b(1S) \to \gamma\gamma)$ will be quite close to the experimental number.

3. Conclusions

We have performed a phenomenological analysis of the NLL results obtained in Ref. [1] for the heavy quarkonium decays to e^+e^- and to two photons.

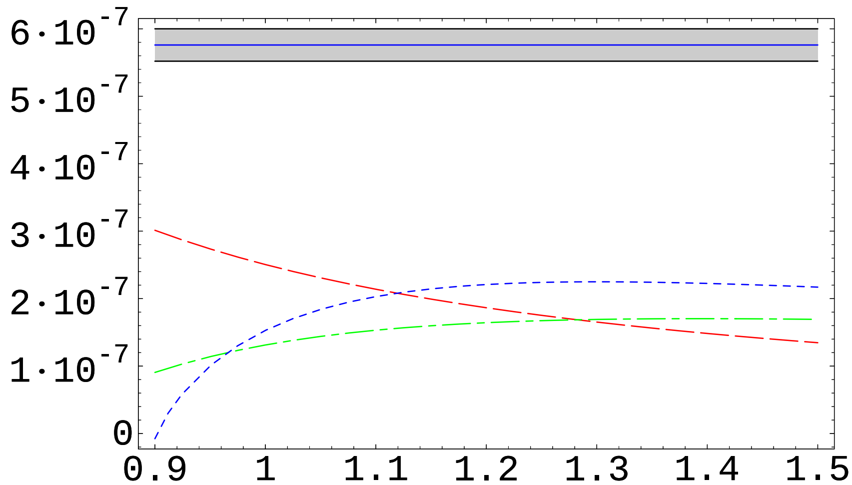

Fig. 5. *Plot of* $\Gamma(\Upsilon(2S) \to e^+e^-)$ *with LO/LL (dashed line), NLO (dot-dashed line) and NLL (dotted line) accuracy versus the renormalization scale* ν*. The horizontal line and its band give the experimental value and its errors:* $\Gamma(\Upsilon(2S) \to e^+e^-) = 0.576 \pm 0.024$ *KeV [6].*

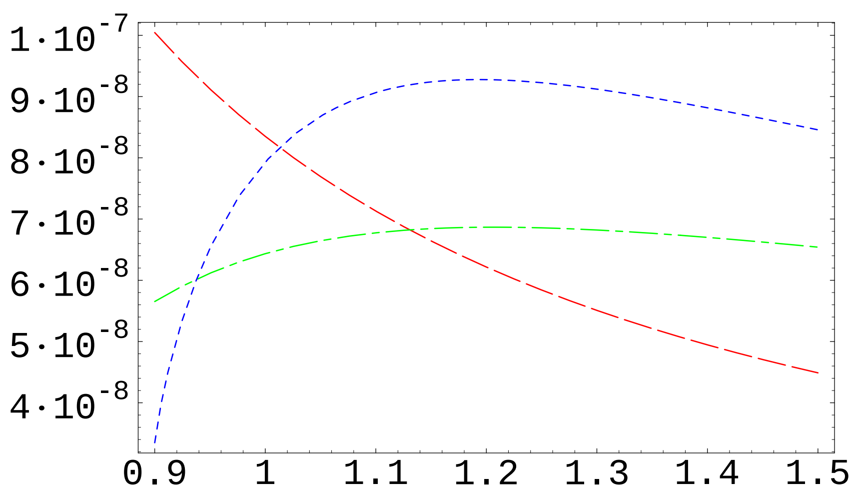

Fig. 6. *Plot of* $\Gamma(\eta_b(2S) \to \gamma\gamma)$ *with LO/LL (dashed line), NLO (dot-dashed line) and NLL (dotted line) accuracy versus the renormalization scale* ν*.*

For $\Gamma(\Upsilon(1S) \to e^+e^-)$, we find that the resummation of logs significantly improves the agreement of the perturbative result with experiment

in relation with a pure NLO evaluation, such that, at the place of minimal sensitive, the theoretical number is off the experimental one by around 50%. Moreover, overall convergence is found, with the NLL order result being a correction with respect the LL order one. Note that in order to be so, we have to take ν of the order of the soft scale such that the large logs are resummed. Surprisingly, the very same picture holds for $\Gamma(J/\Psi(1S) \rightarrow e^+e^-)$ too. It would certainly be challenging to have the complete NNLL result to check whether this pattern of convergence survives, since the difference with experiment is still large in both cases. If this is so one could start to reliable estimate non-perturbative effects in heavy quarkonium (and its relation with the ultrasoft cutoff) and, for the charmonium case, support the view that one can actually use perturbation theory as an starting point for its study (see [7]). On the other hand, for $\Gamma(\Upsilon(2S) \rightarrow e^+e^-)$, we find an strong discrepancy with the experimental figure, rasing doubts that one can actually use perturbation theory there. This could make sense since, for Coulomb-type bound states, the soft scale of the $\Upsilon(2S)$ would be $\sim 1/n^2(= 1/4) \times$ the soft scale of the $\Upsilon(1S)$ (partially corrected by the fact that $\alpha_{\rm s}$ would be larger for the 2S state than for the 1S one). Even if the resummation of logs helps, perturbation theory still breaks down before the renormalization scale ν can reach the typical soft scale of the problem. Somewhat we feel that something similar still happens for the $\Upsilon(1S)$ and the $J/\Psi(1S)$ in a less severe way, since in these last cases one can get much closer to what we believe are the typical soft scales of the problem. In any case, we would not like to draw definite conclusions just from this analysis.

For the decay of the heavy quarkonium to two photons we only have experimental data for the Charmonium case. In this case perfect agreement with experiment is obtained. This boosts our confidence that our number for $\Gamma(\eta_b(1S) \rightarrow \gamma\gamma)$ will be reliable, which we take as one of the major results of this paper.

For $\Gamma(\eta_b(2S) \rightarrow \gamma\gamma)$, we prefer not to draw any conclusion in view of the failure of the results obtained for $\Gamma(\Upsilon(2S) \rightarrow e^+e^-)$.

It is a general trend that the resummation of logs improves the agreement with the experimental result (when available). This can be understood by the fact that makes perturbation theory stable up to smaller scales.

We can also see that the picture is quite similar to the one obtained in Ref. [8] for the hyperfine of the heavy quarkonium, which has recently been computed with NLL accuracy.

The fact that we obtain reasonable numbers for the charmonium system can be considered a surprise. One may think that in this case one has run down to very low scales the ultrasoft scale. Actually, at the numerical level, the ultrasoft scales for the ground states of the bottomonium and charmonium seem to be similar. The reason is that, even if the soft scale

of charmonium is smaller than of bottomonium, this is compensated by the fact that the charm mass is smaller than the bottom mass. This explains the similar behavior found for both systems with errors of the same size. For the $n = 2$ bottomonium states, the behavior seems to be different, with a major breakdown of perturbation theory, making the results not trustworthy (actually a factor two disagreement with experiment is obtained when experimental results are available). In any case, even for the bottomonium and charmonium ground state, the ultrasoft scale has been run down to very low scales. This is a potential problem of the whole analysis. The issue would be certainly clarified if a complete NNLL computation were available for the decays. In case a convergent pattern is observed for the perturbative series, it would certainly be considered an indication that perturbation theory can be applied for these systems. One can then start considering a quantitative study of non-perturbative effects and its relation with the ultrasoft cutoff. In particular one may start to consider what the ratio between $\Lambda_{\rm QCD}$ and mv^2 exactly is and try to apply the results obtained in Ref. [9] for the non-perturbative corrections.

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